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PUNCTE REMARCABILE ÎN TRIUNGHI DISTANŢE. INEGALITĂŢI

DE LA INIŢIERE LA PERFORMANŢĂ

Editura Paralela 45

Cuprins

capitolul 1

Identităţi remarcabile într-un triunghi

„Fiecare om are un orizont. Când acest orizont se îngustează foarte mult, el devine un punct, iar omul exclamă: ***Iată punctul meu de vedere.***"

David Hilbert

§ 1.1. Identităţi remarcabile cu laturi şi raze

$$\sum (p-a)(a+b)(a+c) = p\left(p^2+5r^2+8Rr\right).$$

$$\sum a^2 = 2\left(p^2-r^2-4Rr\right). \qquad \sum a^2 = 2p\left(p^2-3r^2-6Rr\right).$$

$$\sum a^4 = 2\left[p^4-p^2\left(8Rr+6r^2\right)+r^2\left(4R+r\right)^2\right].$$

$$\sum b^2c^2 = p^4+p^2\left(2r^2-8Rr\right)+r^2\left(4R+r\right)^2.$$

$$\sum a^2(p-a)^2 = 2r^2\left[\left(4R+r\right)^2-p^2\right].$$

$$\sum \frac{a(b+c)^2}{bc(p-a)} = \frac{p^2-3r^2}{r^2}. \qquad \sum \frac{(b+c)^2}{b^2c^2(p-a)} = \frac{p^2-3r^2}{4Rr^3p}.$$

$$\sum a^2(b+c)^2(p-b)(p-c) = 4p^2Rr\left(p^2-3r^2\right).$$

$$\sum \frac{a^2(b+c)^2}{p(p-a)} = \frac{4R}{r}\left(p^2-3r^2\right).$$

$$\sum a^2(b+c)(p-b)(p-c) = 2pr\left[p^2(2R-r)-r^2(4R+r)\right].$$

$$\sum \frac{a(b+c)^2}{p(p-a)} = \frac{2}{rp}\left[p^2(2R+r)+r^2(4R+r)\right].$$

$$\sum a(b+c)^2(p-b)(p-c) = 2rp\left[p^2(2R+r)+r^2(4R+r)\right].$$

$$\sum \frac{a^2(b+c)}{p(p-a)} = \frac{2}{rp}\left[p^2(2R-r)-r^2(4R+r)\right].$$

$$\sum\frac{a(p-b)(p-c)}{(a+b)^2(a+c)^2}=\frac{r\left[p^2(2R+r)+r^2(4R+r)\right]}{2p\left(p^2+r^2+2Rr\right)^2}.$$

$$\sum\frac{p-a}{b+c}=\frac{p^2+5r^2+8Rr}{2\left(p^2+r^2+2Rr\right)}. \qquad \sum\frac{p-a}{a}=\frac{p^2+r^2-8Rr}{4Rr}.$$

$$\sum bc\left(b^2+c^2\right)=2p^4-8p^2Rr-2r^2(4R+r)^2.$$

$$\sum bc\left(b^3+c^3\right)=2p\left[p^4-p^2\left(2r^2+4Rr\right)-r^2\left(8R^2+14Rr+3r^2\right)\right].$$

$$\sum\frac{b^2+c^2}{p-a}=\frac{2p^2(2R+3r)-2r(4R+r)^2}{rp}.$$

$$\sum\left(b^2+c^2\right)(p-b)(p-c)=2p^2\left(2Rr+3r^2\right)-2r^2(4R+r)^2.$$

$$\sum(b+c)(p-b)^3(p-c)^3=2r^3p\left[(4R+r)^2(2R+r)-p^2(6R+r)\right].$$

$$\sum\frac{(b+c)(p-b)(p-c)}{(p-a)^2}=\frac{2}{rp}\left[(4R+r)^2(2R+r)-p^2(6R+r)\right].$$

$$\sum bc(b+c)(p-b)(p-c)=2r^2p\left[p^2+(4R+r)(2R+r)\right].$$

$$\sum\frac{b+c}{a(p-a)}=\frac{p^2+(4R+r)(2R+r)}{2Rrp}.$$

$$\sum a(b+c)(p-b)^2(p-c)^2=2r^2p^2\left(8R^2+4Rr-r^2-p^2\right).$$

$$\sum\frac{(b+c)(p-b)(p-c)}{bc(p-a)}=\frac{8R^2+4Rr-r^2-p^2}{2Rr}.$$

$$\sum\frac{(b+c)^2}{bc}=\frac{p^2+r^2+10Rr}{2Rr}. \qquad \sum a(a+b)(a+c)=4p\left(p^2-r^2-Rr\right).$$

$$\sum a^2(a+b)(a+c)=4p^2\left(p^2-3r^2-4Rr\right).$$

$$\sum\frac{a}{bc(b+c)}=\frac{p^2-3r^2-4Rr}{2Rr\left(p^2+r^2+2Rr\right)}.$$

$$\sum\frac{a}{b+c}=\frac{2\left(p^2-r^2-Rr\right)}{p^2+r^2+2Rr}. \qquad \sum\frac{b+c}{a}=\frac{p^2+r^2-2Rr}{2Rr}.$$

$$\sum\left(\frac{a}{b+c}\right)^2=\frac{2\left[p^4-p^2\left(4Rr+6r^2\right)+r^2\left(6R^2+4Rr+r^2\right)\right]}{\left(p^2+r^2+2Rr\right)^2}.$$

$$\sum a^2(a+b)^2(a+c)^2=8p^2\left[p^4-p^2\left(4Rr+6r^2\right)+r^2\left(6R^2+4Rr+r^2\right)\right].$$

$$\sum a\left(b^2+c^2-a^2\right)(p-b)(p-c)=8r^2p\left[p^2-R(4R+r)\right].$$

$$\sum\frac{a^2}{bc(b+c)}=\frac{p^4-6p^2\left(r^2+Rr\right)+r^2\left(8R^2+6Rr+r^2\right)}{2Rrp\left(p^2+r^2+2Rr\right)}.$$

$$\sum a^3(a+b)(a+c)=4p\left[p^4-6p^2\left(r^2+Rr\right)+r^2\left(8R^2+6Rr+r^2\right)\right].$$

$$\sum a(p-a)=2r(4R+r).\qquad \sum a^2(p-a)=4rp(R+r).$$

$$\sum a^3(p-a)=2\left[p^2(2R+3r)-r(4R+r)^2\right].$$

$$\sum a^4(p-a)=4rp\left[p^2(R+2r)-r\left(12R^2+11Rr+2r^2\right)\right].$$

$$\sum a^5(p-a)=p^6-p^4\left(8Rr+23r^2\right)+p^2\left(80R^2r^2+60Rr^3+37r^4\right)-r^3(4R+r)^3.$$

$$\sum a(b+c)=2\left(p^2+r^2+4Rr\right).\qquad \sum a^2(b+c)=2p\left(p^2+r^2-2Rr\right).$$

$$\sum a^3(b+c)=2\left[p^4-4p^2Rr-r^2(4R+r)^2\right].$$

$$\sum a^4(b+c)=2p\left[p^4-p^2\left(6Rr+2r^2\right)-r^2\left(8R^2+14Rr+3r^2\right)\right].$$

$$\sum a^5(b+c)=3p^6-p^4\left(28Rr+43r^2\right)+p^2\left(160R^2r^2+120Rr^3+17r^4\right)-r^3(4R+r)^3.$$

$$\sum\frac{(b+c)^2}{a}=\frac{p\left(p^2+r^2-6Rr\right)}{Rr}.\qquad \sum bc(b+c)^2=4p^2\left(p^2+r^2-6Rr\right).$$

$$\sum\frac{(b+c)^3}{a}=\frac{2\left[p^4+p^2\left(r^2-9Rr\right)+Rr\left(r^2+4Rr\right)\right]}{Rr}.$$

$$\sum bc(b+c)^3=8p\left[p^4+p^2\left(r^2-9Rr\right)+Rr\left(r^2+4Rr\right)\right].$$

$$\sum\frac{(b+c)^2}{bc}=\frac{p^2+r^2+10Rr}{2Rr}.\qquad \sum\frac{(b+c)^3}{bc}=\frac{p^4+20p^2Rr-r^2(4R+r)^2}{2Rrp}.$$

$$\sum a(b+c)^3=2\left[p^4+20p^2Rr-r^2(4R+r)^2\right].$$

$$\sum(b+c)^2=2\left(3p^2-r^2-4Rr\right).$$

$$\sum(b+c)^2(p-b)(p-c)=2\left(3p^2-r^2-4Rr\right).$$

$$\sum\frac{b^2c^2}{(p-b)(p-c)}=\frac{p^4+p^2\left(2r^2-12Rr\right)+r^3(4R+r)}{r^2}.$$

$$\sum b^2c^2(p-a)=p\left[p^4+p^2\left(2r^2-12Rr\right)+r^3(4R+r)\right].$$

$$\sum \frac{a}{(p-b)(p-c)} = \frac{2(4R+r)}{rp}. \qquad \sum \frac{a^2}{(p-b)(p-c)} = \frac{4(R+r)}{r}.$$

$$\sum \frac{a^3}{(p-b)(p-c)} = \frac{2\left[p^2(2R+3r) - r(4R+r)^2\right]}{rp}.$$

$$\sum \frac{a^4}{(p-b)(p-c)} = \frac{4\left[p^2(R+2r) - r\left(12R^2+11Rr+2r^2\right)\right]}{r}.$$

$$\sum \frac{a^5}{(p-b)(p-c)} = \frac{p^6 - p^4\left(8Rr+23r^2\right) + p^2\left(80R^2r^2+60Rr^3+37r^4\right) - r^3(4R+r)^3}{pr^2}.$$

$$\sum \frac{a}{(p-b)^2(p-c)^2} = \frac{2(2R-r)}{r^3p}. \qquad \sum \frac{(p-b)(p-c)}{a} = \frac{rp}{4R}\left[\left(\frac{4R+r}{p}\right)^2 + 1\right].$$

$$\sum \frac{b^2+c^2-a^2}{b+c-a} = \frac{5p^2-(4R+r)^2}{p}. \qquad \sum \frac{b^2+c^2-a^2}{bc} = \frac{2(R+r)}{R}.$$

$$\sum \frac{1}{bc} = \frac{1}{2Rr}. \qquad \sum \frac{b^2+c^2}{bc} = \frac{p^2+r^2-2Rr}{2Rr}.$$

$$\sum \frac{(b+c)^2}{p-a} = \frac{4p(R+2r)}{r}. \qquad \sum \frac{(b+c)^2}{bc} = \frac{p^2+r^2+10Rr}{2Rr}.$$

$$\sum \frac{b^2+c^2-a^2}{a} = \frac{p^4-8p^2Rr-r^2(4R+r)^2}{2Rrp}.$$

$$\sum \frac{b^2+c^2-a^2}{a} = \frac{5p^2-r^2(4R+r)^2}{p}.$$

$$\sum \frac{(p-b)(p-c)}{b+c} = \frac{p^2\left(8Rr-3r^2\right)+r^2(4R+r)^2}{2p\left(p^2+r^2+2Rr\right)}.$$

$$\sum \frac{b+c}{p-a} = 4\left(1+\frac{R}{r}\right). \qquad \sum \frac{p-a}{b+c} = \frac{p^2+5r^2+8Rr}{2\left(p^2+r^2+2Rr\right)}.$$

$$\sum \frac{bc}{b+c}\cos^2\frac{A}{2} = \frac{p\left(p^2+5r^2+8Rr\right)}{2\left(p^2+r^2+2Rr\right)}. \qquad \sum a^2(p-a)^2 = 2r^2\left[(4R+r)^2 - p^2\right].$$

$$\sum bc(b+c) = 2p\left(p^2+r^2-2Rr\right). \qquad \sum \frac{1}{a} = \frac{p^2+r^2+4Rr}{4Rrp}.$$

$$\sum \frac{1}{a^2} = \frac{p^4+p^2\left(2r^2-8Rr\right)+r^2(4R+r)^2}{16R^2r^2p^2}.$$

$$\sum(b+c)^2=2\left(3p^2-r^2-4Rr\right).$$

$$\sum\frac{1}{b+c}=\frac{5p^2+r^2+4Rr}{2p\left(p^2+r^2+2Rr\right)}.$$

$$\sum\frac{bc}{b+c}=\frac{p^4+p^2\left(16Rr+2r^2\right)+r^2\left(4R+r\right)^2}{2p\left(p^2+r^2+2Rr\right)}.$$

$$\sum\frac{a(b+c)}{p-a}=\frac{4Rp}{r}.$$

$$\sum\frac{1}{p-a}=\frac{4R+r}{rp}.$$

$$\sum\frac{p}{p-a}=\frac{2(2R-r)}{r}.$$

$$\sum\frac{1}{a(p-a)}=\frac{p^2+(4R+r)^2}{4Rrp^2}.$$

$$\sum\frac{a}{(p-b)(p-b)}=\frac{2(4R+r)}{rp}.$$

$$\sum\frac{(p-b)(p-c)}{p-a}=\frac{(4R+r)^2-2p^2}{p}.$$

$$\sum\frac{(p-b)(p-c)}{bc}=\frac{2R-r}{2R}.$$

$$\sum\frac{(p-a)^2}{a}=\frac{p\left(p^2+r^2-12Rr\right)}{4Rr}.$$

$$\sum\frac{p-a}{bc}=\frac{4R+r}{2Rp}.$$

$$\sum\frac{p-a}{a^2}=\frac{p^4+p^2\left(2r^2-12Rr\right)+r^3(4R+r)}{16R^2r^2p}.$$

$$\sum a(p-a)=2r(4R+r).$$

$$\sum a(p-a)^2=2rp(2R-r).$$

$$\sum\frac{a}{(p-a)^2}=\frac{4R(4R+r)-2p^2}{r^2p}.$$

$$\sum\frac{a(p-b)(p-c)}{p-a}=4R(4R+r)-2p^2.$$

$$\sum\frac{1}{(p-a)^2}=\frac{(4R+r)^2-2p^2}{r^2p^2}.$$

$$\sum a^2(p-a)^2=2r^2\left[(4R+r)^2-p^2\right].$$

$$\sum\frac{p-a}{b+c}=\frac{p^2+5r^2+8Rr}{2\left(p^2+r^2+2Rr\right)}.$$

$$\sum\frac{2p-a}{2p+a}=\frac{15p^2-r^2-10Rr}{9p^2+r^2+6Rr}.$$

$$\sum(2p-a)(2p+b)(2p+c)=2p\left(15p^2-r^2-10Rr\right).$$

$$\prod(2p+a)=2p\left(9p^2+r^2+6Rr\right).$$

$$\sum\frac{p-a}{p+a}=\frac{4p^2-r^2-16Rr}{4p^2+r^2+8Rr}.$$

$$\sum\frac{p-a}{a}=\frac{p^2+r^2-8Rr}{4Rr}.$$

$$\sum(p-a)^2=p^2-2r^2-8Rr.$$

$$\sum(p-a)^3=p\left(p^2-12Rr\right).$$

$$\sum\frac{p-a}{(p-b)(p-c)}=\frac{p^2-2r^2-8Rr}{r^2p}.$$

$$\sum a^2(p-a)=4rp(R+r).$$

$$\sum\frac{a^2}{p-a}=\frac{4p(R-r)}{r}.$$

$$\sum a^3(p-a)=2r\left[p^2(2R+3r)-r(4R+r)^2\right].$$

$$\sum\frac{a^3}{p-a}=\frac{2p^2(2R-3r)+2r^2(4R+r)}{r}.$$

$$\sum(p-a)^4=p^4-16p^2Rr+2r^2(4R+r)^2.$$

$$\sum\frac{1}{a^2(p-a)}=\frac{p^4+p^2(2r^2-4Rr)+r(4R+r)^3}{16R^2r^2p^3}.$$

$$\sum a^2(p-b)(p-c)=4rp^2(R-r).$$

$$\sum\frac{1}{a(p-a)^2}=\frac{p^2(r-8R)+(4R+r)^3}{4Rr^2p^3}.$$

$$\sum\frac{a(b+c)}{p-a}=\frac{4Rp}{r}.$$

$$\sum\frac{a(b+c)}{(p-b)(p-c)}=\frac{12R}{r}.$$

$$\sum a(b+c)(p-b)(p-c)=4Rrp^2.$$

$$\sum(b+c)(p-b)(p-c)=4rp(R+r).$$

$$\sum\frac{(p-b)(p-c)}{(a+b)(a+c)}=\frac{2r(R+r)}{p^2+r^2+2Rr}.$$

$$\sum a^2(a+b)(a+c)=4p^2\left(p^2-3r^2-4Rr\right).$$

$$\sum bc(b+c)^2=4p^2\left(p^2-3r^2-4Rr\right).$$

$$\sum\frac{(b+c)^2}{a}=\frac{p\left(p^2+r^2-6Rr\right)}{Rr}.$$

$$\sum\frac{a}{bc(b+c)}=\frac{p^2-3r^2-4Rr}{2Rr\left(p^2+r^2+2Rr\right)}.$$

$$\sum b^2c^2(p-a)=p\left[p^4+p^2\left(2r^2-12Rr\right)+r^3(4R+r)\right].$$

$$\sum a(p-a)^3=2r\left[2Rp^2-r(4R+r)^2\right].$$

$$\sum\frac{(p-a)^2}{bc}=1-\frac{r}{2R}.$$

$$\sum\frac{(p-a)^3}{bc}=\frac{2Rp^2-r(4R+r)^2}{2Rp}.$$

$$\sum a^2(p-b)^2(p-c)^2=2r^2p^2\left(8R^2+r^2-p^2\right).$$

$$\sum a^2(p-a)^3=4r^2p\left(4R^2-3Rr-r^2\right).$$

$$\sum\frac{1}{(p-b)^2(p-c)^2}=\frac{p^2-2r^2-8Rr}{r^4p^2}.$$

$$\sum\frac{1}{a(p-b)^2(p-c)^2}=\frac{p^2+r^2-12Rr}{4Rr^5p}.$$

$$\sum\frac{a^2}{(p-a)^2}=\frac{2\left(8R^2+r^2-p^2\right)}{r^2}.$$

$$\sum\frac{(p-a)^2}{b^2c^2}=\frac{(4R+r)^2-p^2}{8R^2p^2}.$$

$$\sum a^3(p-b)(p-c)=2rp\left[p^2(2R-3r)+r^2(4R+r)\right].$$

$$\sum b^2c^2(p-b)(p-c)=r^2\left[p^2\left(p^2+2r^2-4Rr\right)+r(4R+r)^3\right].$$

$$\sum\frac{bc}{a(p-a)}=\frac{p^2\left(p^2+2r^2-4Rr\right)+r(4R+r)^3}{4p^2Rr}.$$

$$\sum\frac{1}{a(p-b)(p-c)}=\frac{p^2+r^2-8Rr}{4Rr^3p}.$$

$$\sum\frac{bc}{p-a}=\frac{p^2+(4R+r)^2}{p}.$$

$$\sum(p-b)(p-c)=r(4R+r).$$

$$\sum\frac{bc}{(p-b)(p-c)}=\frac{p^2+r^2-8Rr}{r^2}.$$

$$\sum\frac{1}{bc(p-b)(p-c)}=\frac{4R+r}{2Rr^2p^2}.$$

$$\sum\frac{1}{(p-b)(p-c)}=\frac{1}{r^2}.$$

$$\sum bc(p-a)=p\left(p^2+r^2-8Rr\right).$$

$$\sum\frac{a^2}{(p-b)(p-c)}=4\left(1+\frac{R}{r}\right).$$

$$\sum a(b+c)(p-a)=12Rrp.$$

$$\sum\frac{1}{a^2(p-b)(p-c)}=\frac{p^4+p^2\left(2r^2-12Rr\right)+r^3(4R+r)}{16p^2R^2r^4}.$$

$$\sum(p-b)^2(p-c)^2=r^2\left[(4R+r)^2-2p^2\right].$$

$$\sum a(p-b)^2(p-c)^2=2r^2p\left(8R^2+2Rr-p^2\right).$$

$$\sum\frac{a^3}{(p-b)(p-c)}=\frac{p^2(2R+3r)-r(4R+r)}{rp}.$$

$$\sum bc(p-b)(p-c)=r^2\left[p^2+(4R+r)^2\right].$$

$$\sum bc(p-a)^2=p^2\left(p^2+r^2-12Rr\right).$$

$$\sum\frac{1}{bc(p-a)}=\frac{2R-r}{2Rr^2p}.$$

$$\sum\frac{(p-a)^2}{bc}=\frac{2R-r}{2R}.$$

$$\sum\frac{p-a}{bc}=\frac{4R+r}{2Rp}.$$

$$\sum\frac{(p-b)(p-c)}{a(p-a)}=\frac{p^2(r-8R)+(4R+r)^3}{4Rp^2}.$$

$$\sum\frac{bc}{p-a}=1+\left(\frac{4R+r}{p}\right)^2.$$

$$\sum a\left(\frac{1}{b}+\frac{1}{c}\right)=\frac{p^2+r^2-2Rr}{2Rr}.$$

$$\sum a(p-a)^4=2S\left[2Rp^2+r\left(r^2-2Rr-24R^2\right)\right].$$

$$\sum\frac{(p-a)^4}{bc}=\frac{2Rp^2+r\left(r^2-2Rr-24R^2\right)}{2R}.$$

$$\sum a^4(p-a)=4S\left[p^2(R+2r)-r\left(2r^2+11Rr+12R^2\right)\right].$$

$$\sum\frac{a^3(p-a)}{bc}=\frac{p^2(R+2r)-r\left(2r^2+11Rr+12R^2\right)}{R}.$$

$$\sum a(p-a)^3=4p^2Rr-2r^2(4R+r)^2.\qquad \sum\frac{(p-a)^3}{bc}=\frac{2p^2R-r(4R+r)^2}{2Rp}.$$

$$\sum\frac{(b+c)(p-a)}{bc}=3.\qquad \sum\frac{p-a}{b^2c^2}=\frac{R+r}{4R^2S}.$$

$$\sum\frac{p-a}{bc\left(b^2+c^2\right)}=\frac{p^6+p^4\left(8Rr-9r^2\right)+p^2r^2\left(112R^2+116Rr+19r^2\right)-3r^3(4R+r)^3}{8Rrp\left[p^6+p^4\left(r^2-12Rr\right)+p^2r^2\left(40R^2+8Rr-r^2\right)-r^3(4R+r)^3\right]}.$$

$$\sum a(p-a)\left(a^2+b^2\right)\left(a^2+c^2\right)=$$

$$=p^6+p^4\left(8Rr-9r^2\right)+p^2r^2\left(112R^2+116Rr+19r^2\right)-3r^3(4R+r)^3.$$

$$\sum a^2\left(2b^2+2c^2-a^2\right)=2\left[p^4+p^2\left(10r^2-8Rr\right)+r^2(4R+r)^2\right].$$

$$\prod\left(2b^2+2c^2-a^2\right)=4\left[p^6+p^4-p^2\left(60R^2r^2+120Rr^3+33r^4\right)-r^3(4R+r)^3\right].$$

$$\sum a^3(a+b)^2(a+c)^2=8p^3\left[p^4-p^2\left(6Rr+10r^2\right)+r^2\left(20R^2+18Rr+5r^2\right)\right].$$

$$\sum\frac{a^3}{(b+c)^2}=\frac{2p\left[p^4-p^2\left(6Rr+10r^2\right)+r^2\left(20R^2+18Rr+5r^2\right)\right]}{\left(p^2+r^2+2Rr\right)^2}.$$

$$\sum\frac{(p-b)(p-c)}{a^2}=\frac{p^4+p^2\left(2r^2-4Rr\right)+r(4R+r)^3}{16R^2p^2}.$$

$$\prod(b+c)=2p\left(p^2+r^2+2Rr\right).$$

$$\sum bc(a+b)(a+c)=p^4+p^2\left(16Rr+2r^2\right)+r^2(4R+r)^2.$$

$$\sum\frac{1}{a(b+c)}=\frac{p^4+p^2\left(16Rr+2r^2\right)+r^2(4R+r)^2}{8Rrp^2\left(p^2+r^2+2Rr\right)}.$$

$$\sum(b+c)^2(a+b)(a+c)=8p^2\left(p^2+r^2+2Rr\right).$$

$$\sum\frac{b(p-b)+c(p-c)}{a(p-a)}=\frac{p^2(r-2R)+(4R+r)^3}{2p^2R}.$$

$$\sum bc(p-b)(p-c)\left[b(p-b)+c(p-c)\right]=2r^3\left[p^2(r-2R)+(4R+r)^3\right].$$