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OLYMPIAD PROBLEMS FROM ALL OVER THE WORLD

VOLUME 3
7th GRADE CONTENT

Cartea Românească
EDUCAȚIONAL

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Chapter I
Problems

1. Let $ABCDE$ be a regular pentagon with center M. A point $P \neq M$ is chosen on the line segment MD. The circumcircle of ABP intersects the line segment AE in A and Q and the line through P perpendicular to CD in P and R.

Prove that AR and QR are of the same length.

STEPHAN WAGNER, AUSTRIAN NMO, 2017

2. Let ABC be an acute triangle. Let H denote its orthocenter and D, E and F the feet of its altitudes from A, B and C, respectively. Let the common point of DF and the altitude through B be P. The line perpendicular to BC through P intersects AB in Q. Furthermore, EQ intersects the altitude through A in N.

Prove that N is the mid-point of AH.

KARL CZAKLER, AUSTRIAN NMO, 2017

3. The diagonals AC and BD of the convex quadrilateral $ABCD$ intersect at point O. The points A_1, B_1, C_1 and D_1 from the segments AO, BO, CO and DO, respectively, are such that $AA_1 = CC_1$ and $BB_1 = DD_1$. Let M be the second intersection point of the circumcircles of ΔAOB and ΔCOD; N be the second intersection point of circumcircles of ΔAOD and ΔBOC; P be the second intersection points of the circumcircles of ΔA_1OB_1 and ΔC_1OD_1 and Q be the second intersecting point of circumcircles of ΔA_1OD_1 and ΔB_1OC_1. Prove that the points M, N, P and Q are concyclic.

ALEKSANDAR IVANOV, BULGARIAN NMO, 2017

4. Consider acute scalene ΔABC with altitudes CD, AE and BF. The points E' and F' are symmetric to E and F with respect to A and B, respectively. Point C_1 on the ray $\overrightarrow{CD}$ is such that $DC_1 = 3CD$. Prove that $\sphericalangle E'C_1F' = \sphericalangle ACB$.

STANISLAV CHOBANOV, BULGARIAN NMO, 2017

5. A square is cut into several rectangles, none of which is a square, so that the sides of each rectangle are parallel to the sides of the square. For each rectangle with sides $a, b, a < b$, compute the ratio $\dfrac{a}{b}$. Prove that sum of these ratios is at least 1.

SINGAPORE SMO, 2017

6. In ΔABC, $AB = AC$, D is a point on the side BC and E is a point on the segment AD. Given that $\sphericalangle BED = \sphericalangle BAC = 2\sphericalangle CBD$, prove that $BD = 2CD$.

SINGAPORE SMO, 2017

7. Let a, b, c be nonzero integers, with 1 as their only positive common divisor, such that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$. Find the number of such triples (a, b, c) with:

$$50 \geq |a| \geq |b| \geq |c| \geq 1.$$

SINGAPORE SMO, 2017

8. In the cyclic quadrilateral $ABCD$, the sides AB, DC meet at Q, the side AD, BC meet at P, M is midpoint of BD. If $\sphericalangle APQ = 90°$, prove that PM is perpendicular to AB.

SINGAPORE SMO, 2017

9. The incircle of ΔABC touches the sides BC, CA, AB at D, E, F, respectively. A circle through A and B encloses ΔABC and intersects the line DE at points P and Q. Prove that the midpoint of AB lies on the circumcircle of ΔPQF.

SINGAPORE SMO, 2017

10. The four digit number $ABCD$ has the property that:

$$ABCD = A \cdot BCD + ABC \cdot D.$$

What is the smallest possible value of $ABCD$?

GORDON LESSELS, IRELAND SHL, 2017

11. The images of the reflection of the circumcentre of triangle ABC in the sides of the triangle are X, Y and Z. Prove that XYZ is congruent to the triangle ABC and corresponding sides are parallel.

JIM LEAHY, IRELAND SHL, 2017

12. Given a point P between the legs of an angle with vertex A. Show, with proof, how to construct a line trough P that intersects the legs of the angle at points B and C so that $|PB| = |PC|$.

JIM LEAHY, IRELAND SHL, 2017

13. Two circles intersect at A and B. A common tangent is drawn to the circles at P and Q. A circle is drawn through P, Q and A and the line AB meets this circle again at C. Join CP and CQ and extend both to meet the given circles at F and B, respectively. Prove that P, Q, F and E lie on the circumference of a circle.

JIM LEAHY, IRELAND SHL, 2017

14. Let O be the circumcenter of an acute triangle ABC and let O_1 and O_2 be circumcenters of triangles OAB and OAC, respectively. The circumcircles of triangles OAB and OAC intersect BC at D ($\neq B$) and E ($\neq C$), respectively and the perpendicular bisector of BC intersects AC at F ($\neq A$). Show that the circumcenter of the triangle ADE lies on AC if and only if the point F lies on the line passing O_1 and O_2.

KOREAN NMO, 2017

15. Let L, M be the midpoints of two sides AB, CD of a convex cyclic quadrilateral $ABCD$. Let E be the intersection of its diagonals AC and BD. Suppose the rays AB and DC intersect at a point F and LM and DE intersect at a point P. Let Q be the foot of the perpendicular on the line segment EM from P. Show that if E is the orthocenter of the triangle FLM, then:

$$\frac{EP^2}{EQ}=\frac{1}{2}\left(\frac{BD^2}{DF}-\frac{BC^2}{CF}\right)$$

KOREAN NMO, 2017

16. Four circles are drawn with the sides of the quadrilateral $ABCD$ as diameters. The two circles passing through A meet again at A', the two circles through B at B', the two circles through C at C' and the two circles through D at D'. Suppose that the points A', B', C' and D' are distinct. Prove that the quadrilateral $A'B'C'D'$ is similar to the quadrilateral $ABCD$.
(*Note:* Two quadrilaterals are *similar* if their corresponding angles are equal to each other *and* their corresponding side lengths are in proportion to each other.)

JIM LEAHY, IRELAND NMO, 2017

17. A line segment B_0B_n is divided into n equal parts at points $B_1, B_2, \ldots, B_{n-1}$ and A is a point such that $\sphericalangle B_0AB_n$ is a right angle. Prove that:

$$\sum_{k=0}^{n}\left|AB_k\right|^2=\sum_{k=0}^{n}\left|B_0B_k\right|^2.$$

JIM LEAHY, IRELAND NMO, 2017

18. Jake has 99 empty bags. An unlimited supply of balls is available, where the weight or each ball is a non-negative integer power of 3. Jake chooses a finite number of balls and distributes them into the bags such that each bag contains the same total weight. If, no matter how the bags have been filled, Jake must have chosen k balls of the same weight, find the largest possible value of k.

MARK FLANAGAN, IRELAND NMO, 2017

19. If $a, b, c \geq 0$, $a+b+c=ab+be+ca$, then:

$$\frac{(1+a)(1+b)}{3(1+a^2)}+\frac{(1+b)(1+c)}{3(1+b^2)}+\frac{(1+c)(1+a)}{3(1+c^2)}\leq 1+a+b+c\,.$$

DANIEL SITARU, RMM, ROMANIA

20. If $a, b, c, d > 0$, $a^2+b^2+c^2+d^2=4$, then:

$$\sum\frac{(2a+b+c)(2b+c+d)(2c+d+a)(2d+a+b)}{(a+b+c+d)^3}\leq 16\,.$$

DANIEL SITARU, RMM, ROMANIA

21. If $a, b, c > 0$, then:

$$\left(\sum a^2b^2\right)\left(\sum a^4b^4\right)\left(\sum \frac{1}{a^2b^2}\right)\left(\sum \frac{1}{a^4b^4}\right) \ge \left(\sum a\right)\left(\sum a^2\right)\left(\sum \frac{1}{a}\right)\left(\sum \frac{1}{a^2}\right).$$

DANIEL SITARU, RMM, ROMANIA

22. If $x, y, z > 0$, then:

$$\frac{(x+y+z)\sum (x+y)^2 + 2(x+y)(y+z)(z+x)}{4(x+y+z)^3} \ge \frac{13}{27}.$$

DANIEL SITARU, RMM, ROMANIA

23. Prove that if $x, y, z > 0$, then:

$$\sqrt{\frac{x}{y}} + 2\sqrt{\frac{y}{z}} + 3\sqrt{\frac{z}{x}} \le \sqrt{6\left(\frac{x}{y} + \frac{2y}{z} + \frac{3z}{x}\right)}.$$

DANIEL SITARU, RMM, ROMANIA

24. If $a, b, c > 0$, $a^2 + b^2 + c^2 = 3$, then:

$$\sum a(a+1)(a+2)(a+3) \ge 72.$$

DANIEL SITARU, RMM, ROMANIA

25. If $a, b, c, d, e > 0, a + b + c + d + e > 5$, then:

$$\sum \frac{b+c+d+e}{2a+b+c+d+e} \ge \frac{10}{3}$$

DANIEL SITARU, RMM, ROMANIA

26. If $a, b, c > 0$, then:

$$\left(4\sqrt{\frac{a}{b}} - \frac{a^2}{b^2}\right) + \left(4\sqrt{\frac{b}{c}} - \frac{b^2}{c^2}\right) + \left(4\sqrt{\frac{c}{a}} - \frac{c^2}{a^2}\right) \le 9$$

DANIEL SITARU, RMM, ROMANIA

27. If $x, y, z > 0; x + y + z = 1$, then:

$$\frac{x^2}{z} + \frac{y^2}{x} + \frac{z^2}{y} + 2\left(\frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y}\right) \ge 3.$$

DANIEL SITARU, RMM, ROMANIA

28. If $x, y, z > 0; x + y + z = 1$, then:

$$\frac{x^2}{z} + \frac{y^2}{x} + \frac{z^2}{y} + \frac{x}{y} + \frac{y}{x} + \frac{y}{z} + \frac{z}{y} + \frac{z}{x} + \frac{x}{z} \ge 7.$$

DANIEL SITARU, RMM, ROMANIA

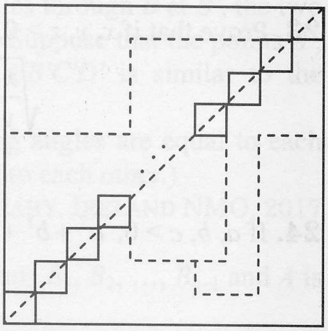
29. Son, his Dad and his Grandfather has ran from their home to a shop and back. Son's velocity was constant, Grandfather's velocity was two times greater than Son's while he was running to the shop and three times less when he was running back. Dad's velocity was two times less than Son's on the way to shop arid 3 times greater when he was running back. Who was the first and who was the last to come home?

UKRAINIAN NMO, 2017

30. There are 22 cards, where the numbers 1, 2, ..., 22 are written. Using this cards, one formed 11 fractions. What is the greatest possible number of integer numbers among the fractions?

UKRAINIAN NMO, 2016

31. Given a checked square. One draw a big diagonal and paint black all the cells such that their centers belong to this diagonal. After the cells on the upper side are cut into two pieces and the lower side is cut into three pieces. It occurred that the areas of these figures are 70, 80, 90 and 100. What is the possible area of the last figure?



BOGDAN RUBLYOV, UKRAINIAN NMO, 2016

32. 30 children – boys and girls – formed a circle. It occurred that there is no child such that both its neighbours are boys. What is the least possible number of girls there?

UKRAINIAN NMO, 2017

33. Let ABC be a triangle. Suppose that AD and BE are its angle bisectors. Prove that $\sphericalangle ACB = 60°$.

DANILO KHILKO, UKRAINIAN NMO, 2016

34. Is it possible to cut a regular triangle into:

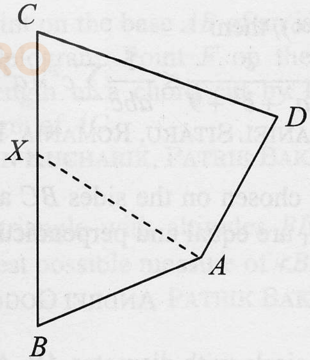
a) three equal quadrilaterals;

b) three equal pentagons?

Convexes of quadrilaterals and pentagons is not obliged.

UKRAINIAN NMO, 2016

35. In the quadrilateral $ABCD$, which is depicted at figure below, the following conditions hold: $\sphericalangle ABC = \sphericalangle BCD$ and $2AB = CD$. The point X is chosen on the side BC, such that $\sphericalangle BAX = \sphericalangle CDA$. Prove that $AX = AD$.



UKRAINIAN NMO, 2016

36. Prove that if $a, b, x, y, z \in (0, \infty)$, then:

$$\frac{yz\left(a^2y+b^2z\right)}{x}+\frac{zx\left(a^2z+b^2x\right)}{y}+\frac{xy\left(a^2x+b^2y\right)}{z}\geq\frac{2}{3}ab\left(x+y+z\right)^2.$$

D.M. BĂTINEȚU-GIURGIU, NECULAI STANCIU, ROMANIA, RMM SUMMER EDITION, 2016

37. Prove that if $a, b, c > 0$; $a + b + c = 3$, then:

$$\sum a\left(\frac{1}{b^3}+\frac{1}{c^3}\right)\geq\frac{18}{a^3+b^3+c^3}.$$

DANIEL SITARU, ROMANIA, RMM SUMMER EDITION, 2016

38. If a, b, c are the length's sides in any triangle, the following relationship doesn't hold:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}=\frac{2}{3}\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right).$$

REDWANE EL MELLAS, MORROCO, RMM SUMMER EDITION, 2016

39. Prove that if $a, b, c \in \mathbb{R}$; $0 < c \leq b \leq a$, then:

$$(a+2b)(a+2c)(b+2c)\leq 8\prod\left(\frac{a^2+ab+b^2}{a+b}\right)\leq(2a+b)(2a+c)(2b+c).$$

DANIEL SITARU, ROMANIA, RMM SUMMER EDITION, 2016

40. Prove that if $a,b,c\in(0,\infty)$; $\sqrt{a}+\sqrt{b}+\sqrt{c}=3$, then:

$$\frac{a\sqrt{b}+b\sqrt{a}}{a-\sqrt{ab}+b}+\frac{b\sqrt{c}+c\sqrt{b}}{b-\sqrt{bc}+c}+\frac{c\sqrt{a}+a\sqrt{c}}{c-\sqrt{ca}+a}\leq 6.$$

DANIEL SITARU, ROMANIA, RMM SUMMER EDITION, 2016

41. Prove that if $a, b, c \in (0, \infty)$, then:

$$12\sum \frac{c}{a^2+b^2+9} \leq \frac{1}{abc}\sum c^2\sqrt{a^2+b^2}.$$

DANIEL SITARU, ROMANIA, RMM SUMMER EDITION, 2016

42. The points A_1 and C_1 are chosen on the sides BC and AB of the triangle ABC so that the segments AA_1 and CC_1 are equal and perpendicular. Prove that if $\measuredangle ABC = 45°$, then $AC = AA_1$.

ANDREI GOGOLEV, UKRAINIAN NMO, 2017

43. A point M is chosen on a circle with diameter AB. A point Qi is also taken on this circle such that $\measuredangle MK_iB < 90°$, where K_i is the intersection point of MQ_i and AB. The chord, which is perpendicular to AB and passes through K_i, intersect BQ_i at P_i. Prove that the points P_i belong to a fixed line, while Q_i vary.

IGOR NAGEL, UKRAINIAN NMO, 2016

44. Let AM be a median in an acute triangle ABC. Its extension intersect the circumcircle w of ABC at P. Let AH_1 be an altitude of ΔABC, H – its orthocenter. The rays MH and PH_1 intersect w at K and T respectively. Prove that the circumcircle of ΔKTH_1 is tangent to BC.

KHILKO DANYLO, UKRAINIAN NMO, 2017

45. Let a, b and c be positive real numbers with $a + b + c = 1$. Prove the inequality:

$$a\sqrt{2b+1}+b\sqrt{2c+1}+c\sqrt{2a+1}\leq\sqrt{2-(a^2+b^2+c^2)}.$$

NIKOLA PETROVIĆ, SERBIAN NMO, 2017

46. Let k be the circumcircle of triangle ABC and let k_a be its excircle opposite to A. The two common tangents of k and k_a meet the line DC at points P and Q. Prove that $\measuredangle PAD = \measuredangle QAC$.

DUŠAN ĐUKIĆ, SERBIAN NMO, 2017

47. Let $ABCD$ be a cyclic quadrilateral. Let O be the circumcenter of the quadrilateral $ABCD$. The diagonals AC and BD intersect at G. Let P, Q, R and S be the circumcenters of triangles AGB, BGC, CGD and DGA respectively. The lines PR and QS intersect at M. Show that M is the midpoint of G and O.

THAILAND NMO, 2017

48. Let ABC be an acute triangle with height AD and $AD = CD$. The median CM intersects it N. Prove that ABC is an isosceles triangle if and only if $CN = 2AM$.

THAILAND NMO, 2017

49. Let ABC be an acute triangle with attitudes AK, BL, CM. Prove that triangle ADC is isosceles if and only if $AM + BK + CL = AL + DM + CK$.

JAROMIR SIMSA, CZECH & SLOVAK NMO, 2017

50. Let D be an arbitrary point on the base AB of an isosceles triangle ABC. Let E be such that $ADEC$ is a parallelogram. Point F on the ray opposite to ED satisfies $EF = EB$. Prove that the length of a chord cut by line BE in the circumcircle of triangle ABE is twice the length of AC.

JAN KUCHARIK, PATRIK BAK, CZECH & SLOVAK NMO, 2017

51. Let ABC be an acute triangle with altitudes BD, CE. Given that $AE \cdot AD = BE \cdot CD$, what is the smallest possible measure of $\sphericalangle BAC$?

PATRIK BAK, CZECH & SLOVAK NMO, 2017

52. If $x, y, z > 0$, then:

$$\left(x^2+2\right)\left(y^2+2\right)\left(z^2+2\right) > 16\sqrt{2xyz\sqrt{xyz}}\ .$$

DANIEL SITARU, RMM, ROMANIA

53. Let $h_1, h_2, h_3, m_1, m_2, m_3$ be the altitudes, respectively the medians of intouch triangle in ΔABC. Prove that:

$$\frac{\dfrac{1}{h_1^2}+\dfrac{1}{h_2^2}+\dfrac{1}{h_3^2}}{\dfrac{1}{h_a^2}+\dfrac{1}{h_b^2}+\dfrac{1}{h_c^2}} = \frac{4R^2}{r^2}\cdot\frac{m_1^2+m_2^2+m_3^2}{m_a^2+m_b^2+m_c^2}$$

MEHMET SAHIN, RMM, TURKEY

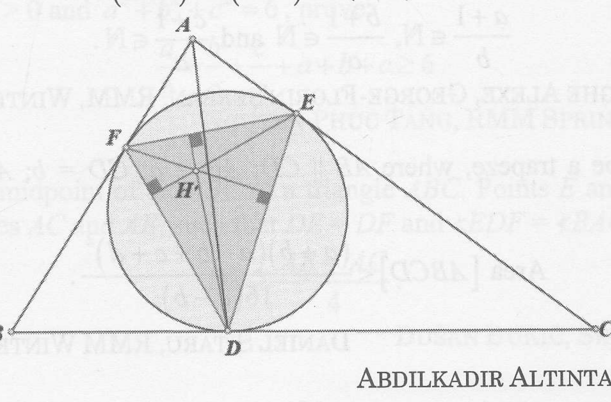
54. Let ΔDEF, $\Delta I_aI_bI_c$ be the contact, respectively the excentral triangle of ΔABC. If $S_1 = S[AFE]$, $S_2 = S[BDF]$, $S_3 = S[CDE]$, $S_0 = S[DEF]$, $A = S[I_aI_bI_c]$, then:

$$4A^2S_1S_2S_3 = S^5,\ S_0A = S^2.$$

MEHMET SAHIN, RMM, TURKEY

55. Prove that:

$$(DH')^2 + (EH')^2 + (FH')^2 \geq 3r^2.$$



ABDILKADIR ALTINTAS, RMM, TURKEY

56. If $a, b > 0$, $a^2 + b^2 = 2$, then:

$$(1+2ab)(2+3ab)(1+3ab) \le 60(2-ab)^3.$$

DANIEL SITARU, RMM, ROMANIA

57. If in ΔABC the nine-point circle and the circumcenter are tangents, then:

$$abc < \frac{8\sqrt{3}}{3}R^3.$$

DANIEL SITARU, RMM, ROMANIA

58. If $x, y, z > 0$, then:

$$\frac{2}{x+y} + \frac{2}{y+z} + \frac{2}{z+x} \le \sqrt{\frac{3(x+y+z)}{xyz}}.$$

DANIEL SITARU, RMM, ROMANIA

59. Let the internal angle bisector of $\measuredangle BAC$ of ΔABC meet side BC at D. Let Γ be the circle through A tangent to BC at D. Suppose Γ meets sides AB and AC at E and F again, respectively. Lines BF and CE meet Γ again at F and Q, respectively. Let AP and AQ intersect side BC at X and Y, respectively. Prove that:

$$XY = \frac{1}{2}BC.$$

HONG KONG, PREIMO, 2017, MOCK EXAM

60. Find all pairs (x, y) of integers satisfying the equation:

$$x^4 - (y+2)x^3 + (y-1)x^2 + (y^2+2)x + y = 2.$$

NGUYEN VIET HUNG, RMM, WINTER EDITION, 2016

61. Find the numbers $a, b, c \in \mathbb{N}^*$, knowing that:

$$\frac{a+1}{b} \in \mathbb{N}, \frac{b+1}{c} \in \mathbb{N} \text{ and } \frac{c+1}{a} \in \mathbb{N}.$$

GHEORGHE ALEXE, GEORGE-FLORIN ȘERBAN, RMM, WINTER EDITION, 2017

62. Let $ABCD$ be a trapeze, where $AB \parallel CD$; $AB = a$; $CD = b$; $AD = c$; $BC = d$; $a > d$. Prove that:

$$\text{Area}\left[ABCD\right] < \frac{(a+b)(a-b+c+d)^2}{16(a-b)}.$$

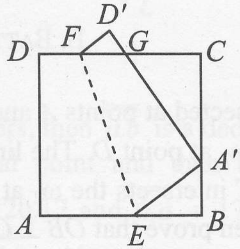
DANIEL SITARU, RMM WINTER EDITION, 2016

63. Let ABC an acute-angled triangle with incentre I. Draw a line to BI at I and let it intersect BC and BA at D and E, respectively. Let P and Q be, respectively, the incentres of the triangles BIA and BIC. Suppose the four points D, E, P, Q are concyclic. Prove that:

$$BA = BC.$$

INDIA, TST, 2017

64. In the given figure, $ABCD$ is a square paper. It is folded along EF such that A goes to a point $A' \neq C, B$ on the side DC and D goes to D'. The line $A'D'$ cuts CD in G. Show that the inradius of the triangle GCA' is the sum of the inradii of the triangles $GD'F$ arid $A'BE$.



INDIAN NMO, 2017

65. Let $ABCDE$ be a convex pentagon in which $\sphericalangle A = \sphericalangle B = \sphericalangle C = \sphericalangle D = 120°$ and whose side lengths are 5 consecutive integers in some order. Find all possible values of $AB + BC + CD$.

INDIAN NMO, 2017

66. Let ABC be a triangle with $\sphericalangle A = 90°$ and $AB < AC$. Let AD be the altitude from A onto BC. Let P, Q and I denote, respectively, the incentres of triangles ABD, ACD and ABC. Prove that:

AI is perpendicular to PQ and $AI = PQ$.

INDIAN NMO, 2017

67. Given $a, b, c > 0$ and $a^2 + b^2 + c^2 = 6$, prove:

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} + a + b + c \geq 6.$$

NGUYEN PHUC TANG, RMM SPRING EDITION, 2017

68. Let D be the midpoint of side BC of a triangle ABC. Points E and F are taken on the respective sides AC and AB, such that $DE = DF$ and $\sphericalangle EDF = \sphericalangle BAC$. Prove that:

$$DE \geq \frac{AB + AC}{4}.$$

DUŠAN ĐUKIĆ, SERBIAN TST, 2017